Outer interval solution of linear systems with parametric interval data

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Abstract— This paper addresses the problem of solving linear systems of equations whose coefficients are depend affine-linearly on parameters varying within prescribed intervals. Such systems, are encountered in many practical problems, e.g in electrical engineering and mechanical systems. A C-XSC(C- for eXtended Scientific Computing) implementation of a symmetric single step method for computing an outer enclosure for the solution set is proposed in this paper. Numerical examples illustrating the applicability of the proposed.

ndex Terms— symmetric single step method, parametric linear systems, validated interval software, C-XSC.

1 INTRODUCTION

IN many practical applications [1], parametric systems involving uncertainties in the parameters have to be solved. In most engineering design problems, linear prediction problems, models in operation research, etc. [2,3] there are usually complicated dependencies between coefficients. The main reason for this dependency is that the errors in several different coefficients may be caused by the same factor [4,5]. More precisely, consider a parametric system

$$A(p)\cdot x = b(p), \qquad (1)$$

where $A(p)\in \Re^{n\times n}$ and $b(p)\in \Re^n$ depend affine linearly on a parameter vector $p\in \Re^k$.

Since, each individual component of $A(p)$, $b(p)$ is an affine-linear combination of the k parameters [6]

$$a_{ij}(p) := a_{ij}^{(0)} + \sum_{\gamma=1}^{k} p_\gamma a_{ij}^{(\gamma)}, b_i(p) := b_i^{(0)} + \sum_{\gamma=1}^{k} p_\gamma b_i^{(\gamma)} \qquad (2)$$

Denote the $k+1$ numerical matrices

$$A^{(0)} := (a_{ij}^{(0)}), A^{(1)} := (a_{ij}^{(1)}), \cdots, A^{(k)} := (a_{ij}^{(k)}) \in \Re^{n\times n}$$

and the corresponding numerical vectors

$$b^{(0)} := (b_i^{(0)}), b^{(1)} := (b_i^{(1)}), \cdots, b^{(k)} := (b_i^{(k)}) \in \Re^{n}.$$

Hence, the parametric matrix and the right-hand side vector can be represented by

$$A(p) := A^{(0)} + \sum_{\gamma=1}^{k} p_\gamma A^{(\gamma)}, b(p) := b^{(0)} + \sum_{\gamma=1}^{k} p_\gamma b^{(\gamma)}$$

and the parametric system (1) can be rewritten in the following form

$$(A^{(0)} + \sum_{\gamma=1}^{k} p_\gamma A^{(\gamma)})x = b^{(0)} + \sum_{\gamma=1}^{k} p_\gamma b^{(\gamma)} \qquad (3)$$

where the parametric vector p varies within the range $[p]\in I\Re^k$.

The solution set of (3), called parametric solution set, and is defined as

$$\begin{aligned}\Sigma^p &:= \Sigma(A(p), b(p), [p]) \\ &:= \{x\in \Re^n | A(p)\cdot x = b(p) \quad \text{for some} \quad p\in [p]\}\end{aligned}$$

Since the solution set has a complicated structure (it does not even need to be convex), which is difficult to find, one looks for the interval hull $\Diamond(\Sigma)$ here Σ is a nonempty bounded subset of $\Re^n$. For $\Sigma\subseteq\Re^n$, define $P\Re^n \to I\Re^n$ by[1]

$$\Diamond(\Sigma) := [\inf\Sigma, \sup\Sigma] = \cap\{ [x]\in I\Re^n \mid \Sigma\subseteq [x] \}.$$

The calculation of $\Diamond(\Sigma)$ is also quite expensive.

Since it is quite expensive to obtain Σ^P or $\Diamond(\Sigma^P)$, it would be a more realistic task to find an interval vector $[y]\in I\Re^n$ which tightly encloses Σ^P ($[y]\supseteq\Diamond(\Sigma^P)\supseteq\Sigma^P$).

Probably the first general purpose method computing outer (and inner) bounds for $\Diamond(\Sigma^P)$ is based on the fixed-point interval iteration theory developed by S. Rump. In [7] Rump applies the general verification theory for system of nonlinear equations for solving parametric linear systems involving affine-linear dependencies. This method was generalized in [8] by proving that a sharp enclosure of the iteration matrix expands the scope of application of the method over problems involving the so-called column-dependent matrices. Meanwhile, there were many attempts to construct suitable methods for solving parameter dependent interval linear systems [1, 3,4,9 ,6,10,11,12]

In practice it is usually required that the matrix $A(p)$ is an H-matrix.

The goal of this paper is to introduce a new C-XSC software (C- for Extended Scientific Computing)[13] for the symmetric single step method by using Interval Centered Form to tightly enclose multivariate nonlinear functions to find the solution set of parametric interval systems, i.e., interval vectors, which contain all possible solutions of this system. We will compare our method to other methods.

The paper is organized as follows. In Section 2 some Basic notations is introduced. In section 3 the dependency problem is presented. The Interval Centered Form is introduced in Section 4. The main results of this paper are presented in Section 5. Another modification for the symmetric single step method is introduced in Section 6. Numerical and practical examples

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[1] $P\Re^n$ is the power set over $\Re^n$. Given a set S, the power set of S is the set off all subset of S

illustrating the features of the proposed method are provided in Section 7. The paper ends with concluding remarks.

2 BASIC NOTATIONS

We use the following notations $\mathfrak{R}, \mathfrak{R}^n, \mathfrak{R}^{n\times n}, I\mathfrak{R}, I\mathfrak{R}^n, I\mathfrak{R}^{n\times n}$, to denote the set of real numbers, the set of real vectors with n components, the set of real $n\times n$ matrices, the set of intervals, the set of interval vectors with n components and the set of $n\times n$ interval matrices, respectively. By interval we mean a real compact interval

$$[x] := [a,b] := \{x \in \mathfrak{R} \mid \ a \le x \le b\}$$

For $[x],[y] := [c,d] \in I\mathfrak{R}$ we define

- The mid-point $\mathrm{mid}([x]) := (a+b)/2$,
- the Radius $\mathrm{rad}([x]) := (b-a)/2$,
- the absolute value $|[x]| := \max\{\ |a|,|b|\ \}$,
- the distance $q([x],[y]) := \max\{|a-c|,|b-d|\}$,
- minimal absolute value (mignitude)

$$\prec [x] \succ := \min\{|x| : x \in [x]\} = \begin{cases} \min\{|a|,|b|\} & \text{if} \quad 0 \notin [x] \\ 0 & \text{else} \end{cases} \qquad (4)$$

For interval vectors and interval matrices, these quantities are defined componentwise.

If for two interval vectors $[u],[v] \in I\mathfrak{R}^n$ we have $[u_i] \cap [v_i] \neq \phi, i = 1,2,\cdots,n,$ then $[u] \cap [v] := ([u_i] \cap [v_i])$, otherwise $[u] \cap [v] := \phi$. In addition, for $[u],[v] \in IR^n$ we define $[u] \subseteq [v]$ iff $[u_i] \subseteq [v_i], i = 1,2,\cdots,n$. Furthermore, we repeat some relations concerning the distance:

$$q([u],[v]) \le q([u],[w]) + q([w],[v]),$$
$$q([u]+[w],[v]+[w]) = q([u],[v]),$$
$$q([u]+[v],[w]+[z]) \le q([u],[w]) + q([v],[z]),$$
$$\text{if} \quad [u],[v],[w],[z] \in I\mathfrak{R}^n$$

For square interval matrices we define the comparison matrix (Ostrowsky matrix)

$\prec [A] \succ := (c_{ij}) \in \mathfrak{R}^{n\times n}$ using (4) by setting

$$c_{ij} := \begin{cases} -|a_{ij}| & \text{if} \quad i \neq j, \\ \prec [a_{ij}] \succ & \text{if} \quad i = j. \end{cases}$$

A square matrix $[A] \in I\mathfrak{R}^{n\times n}$ is called regular if all $A \in [A]$ are nonsingular.

If $\mathrm{mid}([A]) \cdot [A]$ is regular then $[A]$ is strongly regular. An interval matrix $[A]$ is an H-matrix iff there exist a vector $v > 0$ such that $\prec [A] \succ v > 0$.

Definition 1 [14] *Let $A, B, C \in \mathfrak{R}^{n\times n}$. Then $A = B - C$ is a regular splitting of A if $C \ge 0$ and B is nonsingular with $B^{-1} \ge 0$.*

Theorem 1 [14] *Assume that $A \in \mathfrak{R}^{n\times n}$ is nonsingular, that $A^{-1} \ge 0$ and that $A = B - C$ is a regular splitting of A. Then $\rho(B^{-1}C) < 1$, where $\rho(\ )$ denotes the spectral radius of a matrix.*

Regular splitting was introduced in [14], where one can also find the proof of Theorem 1.

3 DEPENDENCY PROBLEMS

The dependency problem arises when one or several variables occur more than once in an interval expression. Dependency may lead to catastrophic overestimation in interval computations. For example, if the interval $[x] = [1,2]$ is subtracted from itself $[x] - [x] = [1,2] - [1,2] = [-1,1]$ is obtained as the result and not the interval [0,0] as expected. Actually, interval arithmetic cannot recognize the multiple occurrence of the same variable$[x]$. The result is $\{x - y | x \in [x], y \in [y]\}$ instead of$\{x - x \mid x \in [x]\}$. In general, when a given variable occurs more than once in an interval computation, it is treated as a different variable in each occurrence.

For a less extreme example, take $f(x) = (5 + x) \cdot (5 - x)$ for $x \in [x] = [-1,1]$. Using the basic formulas of standard interval arithmetic[15,16], we get

$$5 + [x] = [4,6]$$
$$5 - [x] = [4,6]$$
$$(5 + [x]) \cdot (5 - [x]) = [6,16]$$

The interval formulas give an interval whose diameter is 20, whereas the exact interval result f([x])= [24, 25] has a diameter of only 1. Note that when one operand in the product $(5 + x) \cdot (5 - x)$ is at the maximum value 6, the other must be at the minimum value 4; the combination 4· 4 and 6·6, which gave the extreme values of F ([x]), never occur.

A simple remedy for this example is to rewrite (5 + x) · (5-x) = 25 – x^2 , which has only one occurrence of the variable x. An interval computation of this new expression will give the exact result. Unfortunately, this remedy is often impossible to apply in practice.

Several other methods have been proposed to attack the dependency problem. The main class of methods is known as generalized interval arithmetic [17,18], in several incarnations and generalizations, such as mean-value form [2] and slopes [19]. The purpose of Interval Centered Form [20] is to reduce the effect of the dependency problem when computing with standard interval arithmetic.

4 INTERVAL CENTERED FORM

Given any real rational expression $f(p_1, p_2, \cdots, p_k)$ and any vector of real numbers $c = (c_1, c_2, \cdots, c_k)$ at which the value of f is defined (i.e., not at a singularity of f), we can write f as [16,19]

$$f(p_1, p_2, \cdots, p_k) = f(c_1, c_2, \cdots, c_k) + g(p_1 - c_1, p_2 - c_2, \cdots, p_k - c_k) \qquad (5)$$

by substituting $p_i = c_i + u_i, i = 1,2,\cdots,k$ into the expression for $f(p_1, p_2, \cdots, p_k)$ and forming

$$g(p_1 - c_1, p_2 - c_2, \cdots, p_k - c_k) = g(u_1, u_2, \cdots, u_k)$$
$$= f(u_1 + c_1, u_2 + c_2, \cdots, u_k + c_k) - f(c_1, c_2, \cdots, c_k) \qquad (6)$$

By using (6) to form the expression for g in (5), we can substitute interval variables $[u_1], \cdots, [u_k]$ for the real variables $u_1, \cdots, u_k$ occurring in the expression for $g(u_1, \cdots, u_k)$ and obtian an expression we can denote by $g([u_1], \cdots, [u_k])$; interpreting the arithmetic operations as interval arithmetic operations, the resulting expression $g([u_1], \cdots, [u_k])$ is a rational interval expression.

We form the expression

$$F([p_1], [p_2], \cdots, [p_k]) = f(\mathrm{mid}([p_1]), \mathrm{mid}([p_2]), \cdots, \mathrm{mid}([p_k])) + g([p_1] - \mathrm{mid}([p_1]), [p_2] - \mathrm{mid}([p_2]), \cdots, [p_k] - \mathrm{mid}([p_k])) \qquad (7)$$

corresponding to (5) by substituting $[p_i] - \mathrm{mid}([p_i])$ for $[u_i]$ in the rational interval expression $g([u_1], \cdots, [u_k])$.

The equation (7) then, gives an Interval Centered Form of a rational

interval expression with real restriction $f(u_1,\cdots,u_k)$

5 ITERATIVE METHOD

In this section we assume that the reader is familiar with the concept P contractions for proving the convergence of a fixed point iteration to a unique fixed point for an arbitrary starting vector. For the details please see [15,21].

5.1 The symmetric single step method

We assume throughout that the matrix $A(p)$ is nonsingular, and moreover that its diagonal entries $a_{ii}(p)$ are all nonzero. We can express the matrix $A(p)$ as the matrix sum [14,15]

$$A(p)=D(p)+L(p)+U(p), \tag{8}$$

Where $D(p)$ is a diagonal matrix, and $L(p)$ and $U(p)$ are respectively strictly lower and upper triangular matrices. We can write (1) as

$$D(p)\cdot x=b(p)-(L(p)+U(p))\cdot x \tag{9}$$

Then

$$x=D^{-1}(p)\cdot(b(p)-(L(p)+U(p)\cdot x), \tag{10}$$

where

$$L(p)=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{21}(p) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1}(p) & \cdots & a_{nn-1}(p) & 0 \end{pmatrix} \tag{11}$$

$$U(p)=\begin{pmatrix} 0 & a_{12}(p) & \cdots & a_{1n}(p) \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1n}(p) \\ 0 & 0 & \dots & 0 \end{pmatrix} \tag{12}$$

$$D^{-1}(p)=\begin{pmatrix} \frac{1}{a_{11}(p)} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}(p)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{a_{nn}(p)} \end{pmatrix} \tag{13}$$

Since the diagonal entries $a_{ii}(p)$ of $A(p)$ are nonzero, we can carry out the following iterative method derived from (9):

$$\left[x_i^{(l+1/2)}\right]=\left(b_i(p)-\sum_{j=1}^{i-1}a_{ij}(p)\left[x_j^{(l+1/2)}\right]-\sum_{j=i+1}^{n}a_{ij}(p)\left[x_j^{l}\right]\right)/a_{ii}(p),\qquad 1\le i\le n, \tag{14}$$

$$\left[x_i^{(l+1)}\right]=\left(b_i(p)-\sum_{j=1}^{i-1}a_{ij}(p)\left[x_j^{(l+1/2)}\right]-\sum_{j=i+1}^{n}a_{ij}(p)\left[x_j^{l+1}\right]\right)/a_{ii}(p),1\le i\le n,l\ge 0 \tag{15}$$

where the $[x^{(0)}]'s$ initial interval vector. We call this iteration procedure the symmetric single step method. By using (2) we can rewritten (14) and (15) in the following form:

$$\left[x_i^{(l+1/2)}\right]=\frac{\left(b_i^{(0)}-\sum_{j=1}^{i-1}a_{ij}^{(0)}\left[x_j^{(l+1/2)}\right]-\sum_{j=i+1}^{n}a_{ij}^{(0)}\left[x_j^{(l)}\right]\right)}{a_{ii}^{(0)}+\sum_{\gamma=1}^{k}p_\gamma a_{ii}^{(\gamma)}}+\frac{\sum_{\gamma=1}^{k}p_\gamma\left(b_i^{(\gamma)}-\sum_{j=1}^{i-1}a_{ij}^{(\gamma)}\left[x_j^{(l+1/2)}\right]\right)}{a_{ii}^{(0)}+\sum_{\gamma=1}^{k}p_\gamma a_{ii}^{(\gamma)}}\\ \frac{-\sum_{j=i+1}^{n}a_{ij}^{(\gamma)}\left[x_j^{(l)}\right]\Big)}{},\qquad 1\le i\le n, \tag{16}$$

$$\left[x_i^{(l+1}\right]=\frac{\left(b_i^{(0)}-\sum_{j=1}^{i-1}a_{ij}^{(0)}\left[x_j^{(l+1/2)}\right]-\sum_{j=i+1}^{n}a_{ij}^{(0)}\left[x_j^{(l+1)}\right]\right)}{a_{ii}^{(0)}+\sum_{\gamma=1}^{k}p_\gamma a_{ii}^{(\gamma)}}+\frac{\sum_{\gamma=1}^{k}p_\gamma\left(b_i^{(\gamma)}-\sum_{j=1}^{i-1}a_{ij}^{(\gamma)}\left[x_j^{(l+1/2)}\right]\right)}{a_{ii}^{(0)}+\sum_{\gamma=1}^{k}p_\gamma a_{ii}^{(\gamma)}}\qquad l\ge 0\\ \frac{-\sum_{j=i+1}^{n}a_{ij}^{(\gamma)}\left[x_j^{(l+1)}\right]\Big)}{},\qquad 1\le i\le n,\quad p\in[p] \tag{17}$$

5.2 The symmetric single step method with interval centered form

If we have a deep look in the equations (16) and (17), we can find that the parametric $p_\gamma,(\gamma=1,2,\cdots,k)$ occurs more than once (between the numerator and and denominator). In general, when a given variable occurs more than once in an interval computation, it is treated as a different variable in each occurrence. This causes widening of computed intervals and makes it more difficult to obtain sharp result in calculations. One should always be aware of this consideration and take appropriate steps to reduce its effect. We have discussed a way to do this in last section called Interval Centered Form.
In our case, from (16) and (17), the rational function will be in the form:

$$f(p_1,p_2,\cdots,p_k)=\frac{\sum_{i=0}^{k}[\alpha_i]p_i}{\sum_{j=0}^{k}\beta_j p_j} \tag{18}$$

where [α] and β are interval vector and real vector respectively, and $p_0=1$. Then the interval centered form of this function will be in the following form:

$$\frac{\sum_{i=0}^{k}[\alpha_i]p_i}{\sum_{j=0}^{k}\beta_j p_j}=\frac{\sum_{i=0}^{k}[\alpha_i]\cdot\mathrm{mid}([p_i])}{\sum_{j=0}^{k}\beta_j\cdot\mathrm{mid}([p_j])}+\frac{\sum_{i=0}^{k}\sum_{j=1}^{k}(\beta_i[\alpha_j]-\beta_j[\alpha_i])\mathrm{mid}([p_i])\cdot\mathrm{rad}([p_j])}{\sum_{j=0}^{k}\beta_j\cdot\mathrm{mid}([p_j])(\sum_{j=0}^{k}\beta_j\cdot\mathrm{mid}([p_j])+\sum_{i=1}^{k}\beta_i\cdot\mathrm{rad}([p_j]))} \tag{19}$$

By using the equation (19) in the equations (16) and (17), we can get better results.

Theorem 2. *Consider parametric linear system (1), where* $A(p)$ *and* $b(p)$ *are defined by*

$$a_{ij}(p):=a_{ij}^{(0)}+\sum_{\nu=1}^{k}p_\nu a_{ij}^{(\nu)},$$

$$b_i(p):=b_i^{(0)}+\sum_{\nu=1}^{k}p_\nu b_i^{(\nu)},\qquad (i,j=1,2,\cdots,n)$$

We define $L(p)$, $U(p)$ *and* $D^{-1}(p)$ *as in (11), (12) and (13), respectively. Then, the sequence* $\left[x^{(l)}\right]_{l=0}^{\infty}$ *calculated according to the iteration method (symmetric single step method defined as in (16) and (17)), converges for all interval vectors* $[x^{(0)}]\in I\Re^n$ $[x^*]$, *where* $[x^*]$ *is the unique fixed point of the equation (10).*

6 MODIFICATION

In this section we consider modifications of the preceding iterative methods which are based on the fact that if for any of these methods one is starting with an interval vector containing the limit, then all iterates contain the limit. Therefore the enclosure of the limit might be improved by forming intersections after each iteration step.

Symmetric single step method with intersection

Theorem 4. *Let* $A(p) \in \Re^{n\times n}$ *and* $b(p) \in \Re^{n\times n}$ *be given. We define L(p), U(p) and* $D^{-1}(p)$ *as in (11), (12) and (13), respectively. Let* [x*] *is the unique fixed point of the equation (10). We assume that we have an interval vector* [initial] $\in I\Re^n$ *satisfying* [x*]⊆[initial]. *We consider the Symmetric single step method with intersection.*

$$\begin{cases} [x^{(0)}] := [\text{initial}] \\ \text{for} \quad i=1 \quad \text{to} \quad n \quad \text{do} \\ \quad [x_i^{(l+1/2)}] := \\ \quad [x^{(l)}] \cap \{(b_i(p) - \sum_{j=1}^{i-1} a_{ij}(p)[x_j^{(l+1/2)}] \\ \qquad - \sum_{j=i+1}^{n} a_{ij}(p)[x_j^{(l)}])/a_{ii}(p)\} \\ \text{for} \quad i=n \quad \text{down to} \quad 1 \quad \text{do} \\ \quad [x_i^{(l+1)}] := \\ [x_i^{(l+1/2)}] \cap \{(b_i(p) - \sum_{j=1}^{i-1} a_{ij}(p)[x_j^{(l+1/2)}] \\ \qquad - \sum_{j=i+1}^{n} a_{ij}(p)[x_j^{(l+1)}])/a_{ii}(p)\} \end{cases} \qquad (20)$$

Then $\lim_{l\to\infty}[x^{(l)}] = [x^*]$

To get an interval vector [initial]. We assume that $A(p) \in \Re^{n\times n}$ and that $b(p) \in \Re^{n\times n}$ is an H-matrix. Let $D^{-1}(p)$, $L(p)$, $U(p)$ and [x*] defined as in Theorem 2. Then we consider symmetric single step method with arbitrary [x(0)].

We assume that

$P := \prec[D([p])]\succ^{-1}|[L([p])]+[U([p])]|$, where $\rho(P)<1$ (see Theorem 11.4 in [1]).

For $m > l$ we get [2]:

$$\begin{aligned} q([x^{(m)}],[x^{(l)}]) &\le q([x^{(m)}],[x^{(m-1)}]) + \cdots + q([x^{(l+1)}],[x^{(l)}]) \\ &\le P^{m-1}\cdot q([x^{(m)}],[x^{(0)}]) + \cdots + P^{l}\cdot q([x^{(1)}],[x^{(0)}]) \\ &= P^{l}\cdot(I + P + \cdots + P^{m-l-1})\cdot P^{m-1}\cdot q([x^{(1)}],[x^{(0)}]) \\ &\le P^{l}\cdot(\sum_{i=0}^{\infty} P^{i})\cdot P^{m-1}\cdot q([x^{(1)}],[x^{(0)}]) \\ &= P^{l}\cdot(I-P)^{-1}\cdot P^{m-1}\cdot q([x^{(1)}],[x^{(0)}]). \end{aligned}$$

Since $\lim_{m\to\infty}[x^{(m)}] = [x^*]$, it holds that (set $l := 1$)

$$q([x^*],[x^{(l)}]) \le P\cdot(I-P)^{-1}\cdot q([x^{(1)}],[x^{(0)}]) =: u,$$

Then

$$\underline{x}^{(l)} - u \le \underline{x}^*, \quad \overline{x}^* \le \overline{x}^{(l)} + u$$

Hence, we get $[x^*] \subseteq [\underline{x}^{(l)} - u \ , \ \overline{x}^{(l)} + u] =: [\text{initial}]$

Algorithm 1. Parametric interval linear systems (H-Matrix)

1. Computation of an initial interval vector

$P := \prec[D([p])]\succ^{-1}|[L([p])]+[U([p])]|$,

$[\text{initial}] := [\underline{x}^{(l)} - u, \overline{x}^{(l)} + u], \quad l \ge 0$

2. Verification step

[x(1)] := [initial]

repeat

if intersection = 0 **then**

Using equations ((16) and (17)) with (19)

else Using equation (20) with (19)

until $[x^{(l+1)}]$ *and* $[x^{(l)}]$ *are equals*

6.

if $[x^{(l+1)}]$ *and* $[x^{(l)}]$ *are equals* **then**

$\hat{x} \in [x^{(l+1)}]$ ($\hat{x}$ the exact solution)

else *no inclusion can be computed*

7 NUMERICAL AND PRACTICAL EXAMPLES

Example(1):

$$\begin{pmatrix} 1 & p_1 \\ p_1 & p_2 \end{pmatrix}\cdot x = \begin{pmatrix} 2+p_2 \\ 2+p_2 \end{pmatrix},$$

where: $p_1 \in [-0.6,-0.4]$, $p_2 \in [1.8, 2.2]$

iteration =9,

Proposed method	Elaraby [22]	Popova [4]
[4.808641,6.833334]	[4.843137,7.000000]	[4.877162,6.551409]
[2.521604,4.388889]	[2.607843,4.666667]	[2.598498,4.258645]

Example(2):

$$\begin{pmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{pmatrix}\cdot x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

where: $p \in [0, 1]$

iteration =2,

Proposed method	Elaraby [22]	Popova [4]
[0.111111,0.333334]	[0.111111,0.333334]	[0.177533,0.772466]
[0.111111,0.333334]	[0.111111,0.333334]	[0.080561,0.469439]
[0.111111,0.333334]	[0.111111,0.333334]	[-0.382168,0.132168]

N.B . We got the same result as Elaraby, because there is no dependency will be happened between the parameters.

Example(3):

$$\begin{pmatrix} 1+p_1+p_2 & p_1 & p_2 \\ 0 & p_1+p_2 & p_2 \\ 0.1 & 0 & 3p_1+p_2 \end{pmatrix}\cdot x = \begin{pmatrix} p_1+5p_2 \\ 2+p_1+3p_2 \\ 1+2p_1+p_2 \end{pmatrix}$$

where: $p_1 \in [0.4, 0.5]$, $p_2 \in [0.2, 0.3]$

iteration =10,

Proposed method	Elaraby [22]	Popova [4]
[−0.440114,−0.056444]	[-0.776142,-0.132614]	[-0.468005,-0.022753]
[3.604649,4.597432]	[3.325426,5.114795]	[3.631414,5.540667]
[1.258164,1.460008]	[1.103743,1.69829]	[1.256884,1.461288]

Application:

we consider a linear resistive network, presented in [23,24]. The resistive network consists of two current sources J_1 and J_2 and nine resistors. The problem of finding the voltages $v_1,\cdots,v_5$, when the voltage of each conductance $g_i, i=1,2,\cdots,9$ varies independently in prescribed bounds $[g]_i, i=1,2,\cdots,9$, leads to the following parameterized linear system

$$\begin{pmatrix} g_1+g_6 & -g_6 & 0 & 0 & 0 \\ -g_6 & g_2+g_6+g_7 & -g_7 & 0 & 0 \\ 0 & -g_7 & g_3+g_7+g_8 & -g_8 & 0 \\ 0 & 0 & -g_8 & g_4+g_8+g_9 & -g_9 \\ 0 & 0 & 0 & -g_9 & g_5+g_9 \end{pmatrix} \cdot v = J$$

where $J=(10,0,10,0,0)^T$ and the parameters are subject to tolerances $[g]_i=[1-\delta,1+\delta], i=1,2,\cdots,9\cdot$

We solve the system for different values of the tolerances δ varying from 0.1% to 10% of the nominal value.

iteration =24, tolerance=0.1%

Proposed method	HBR method [23]	Elaraby [22]	Popova [3]
[7.01793,7.16388]	[6.8693, 7.2950]	[6.89898,7.29765]	[7.01337,7.16844]
[4.13776,4.22587]	[4.0689, 4.4971]	[3.97569,4.40530]	[4.11566,4.24797]
[5.39797,5.51111]	[5.2501, 5.6612]	[5.26906,5.65655]	[5.39177,5.51732]
[2.15854,2.20509]	[2.0183, 2.3568]	[2.04981,2.32733]	[2.13647,2.22716]
[1.07903,1.10278]	[1.0397, 1.1931]	[1.00461,1.18717]	[1.05937,1.2244]

8 Conclusion

The problem of solving parametric linear systems of equations is very important in practical applications. A simple method for determining an outer solution to the linear system considered has been suggested in section 5 by using the method presented in section 4. Some numerical and practical examples are solved. The methods that presented can be applied to big real life problems such as structural engineering [3,12] without any problems.

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